

Relaying phase synchrony in chaotic oscillator chains

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We study the manner in which the effect of an external drive is transmitted through mutually coupled response systems by examining the phase synchrony between the drive and the response. Two different coupling schemes are used. Homogeneous couplings are via the same variables, while heterogeneous couplings are through different variables. With the latter scenario, synchronization regimes are truncated with increasing number of mutually coupled oscillators, in contrast to homogeneous coupling schemes. Our results are illustrated for systems of coupled chaotic Rössler oscillators.

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I. INTRODUCTION

The many flavours of synchrony that emerge as a consequence of different coupling scenarios have been examined in detail over the last couple of decades. The roles of topology and nonlinearity have been studied, and a fair understanding of the different forms of synchrony that can arise—given a specific scheme through which dynamical systems interact with each other—is understood to some degree [1]. Synchronization in various forms is common in forced and coupled nonlinear systems [2–7]. The manner in which the interacting subsystems are coupled plays a crucial role in determining which form of synchrony arises. For example, when identical nonlinear systems are coupled unidirectionally with one subsystem (the master) driving the response subsystem (the slave) complete synchronization occurs. When the coupled systems are not identical, generalized synchronization can occur [6, 8]. “Mixed” synchrony is observed in counter-rotating coupled oscillators [9] while phase synchronization occurs in mutually coupled chaotic systems: the phases of interacting systems are entrained while the amplitudes remain uncorrelated [7]. This is a phenomenon of great interest due to potential applications in different fields, ranging from physics, chemistry to biological and medical sciences [10].

Since the coupling can be uni- or bi-directional and can be linear or nonlinear [11], and can be through similar or dissimilar variables [12, 13], as the number of interacting components increases, the possible variations grow exponentially. Our interest in present work is to examine a small number of “typical” patterns or motifs of coupling, and investigate the different patterns of synchronization phenomena that result. An additional motivation arises from the fact that in a variety of natural systems that are subject to forcing, the modulation can be either direct, namely when a given system is itself subject to driving, or indirect, when it is coupled to another

system which is the one that is being modulated. Such indirect modulation is likely to be operative in biological phenomena [14] or in networks of coupled dynamical systems [15].

In the present paper we study coupled chaotic oscillators which are externally forced. We find that as the number of mutually coupled oscillators increase, the phase synchronization (PS) regime gets truncated if the coupling is heterogeneous, namely through different variables, while the same does not hold if the coupling is homogeneous, namely through the same variables.

The different coupling schemes are discussed in the following Section II where we also study phase synchronization between the drive and the response. The effect of the drive is further examined in Section III. The measure we used to determine phase synchronization is based upon the variation in phase difference with time: this is discussed in an appendix that follows the concluding Section IV which presents a discussion and summary.

II. THE COUPLING PATTERNS

We first consider the model system of three oscillators coupled as schematically shown in Fig. 1. Oscillators O_2 and O_3 (denoted by the subscripts in the variables) are diffusively and mutually coupled, and are driven by oscillator O_1 . The response oscillators O_2 and O_3 are identical and are distinct from oscillator O_1 , there is a parameter mismatch in the frequencies. The scheme in Fig. 1(a) is termed “heterogeneous” since the driving is effected through a variable that is not involved in the coupling, while that in Fig. 1(b) is termed “homogeneous” since the driven and coupled variables are the same.

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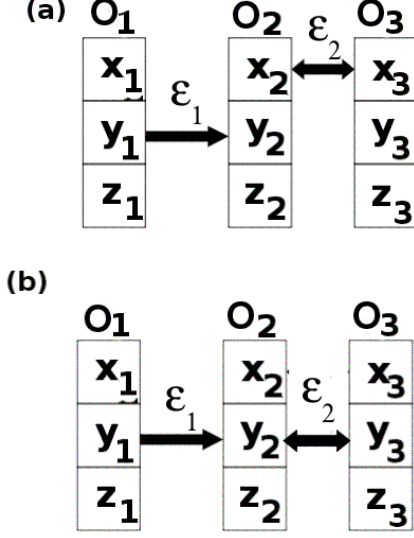


FIG. 1: (a) Heterogeneous coupling: the (nonidentical) chaotic oscillator O_1 is coupled to two mutually coupled identical oscillators through one of the variables, here y , while the identical oscillators O_2 and O_3 are symmetrically and bidirectionally coupled through the x variable. (b) Homogeneous coupling: all the interactions are through the same variable, here y .

A. Heterogeneous Coupling

Consider the system of three coupled Rössler chaotic attractors,

$$\begin{aligned}
 \dot{x}_1 &= -y_1 - z_1 \\
 \dot{y}_1 &= x_1 + a_1 y_1 \\
 \dot{z}_1 &= b_1 + z_1(x_1 - c_1) \\
 \\
 \dot{x}_2 &= -y_2 - z_2 + \epsilon_2(x_3 - x_2) \\
 \dot{y}_2 &= x_2 + a_2 y_2 + \epsilon_1(y_1 - y_2) \\
 \dot{z}_2 &= b_2 + z_2(x_2 - c_2) \\
 \\
 \dot{x}_3 &= -y_3 - z_3 + \epsilon_2(x_2 - x_3) \\
 \dot{y}_3 &= x_3 + a_2 y_3 \\
 \dot{z}_3 &= b_2 + z_3(x_3 - c_2).
 \end{aligned} \tag{1}$$

We study the different synchronization states in coupling parameter's space, ϵ_1 and ϵ_2 . The internal parameters of these oscillators are fixed at $a_1=0.2$, $b_1=0.2$, and $c_1=5.7$ (for driving oscillator O_1); while $a_2=0.15$, $b_2=0.2$, and $c_2=10$ (for mutually coupled oscillators, O_2 and O_3). At these set of parameters all oscillators show chaotic motion. The natural frequencies f_i turn out to be $f_1 = 1.079$ and $f_2 = f_3 = 1.041$ respectively.

Shown in Fig. 2 are schematic phase diagrams with regard to phase synchrony as a function of $\epsilon_1 - \epsilon_2$. The different regimes are characterized through a measure

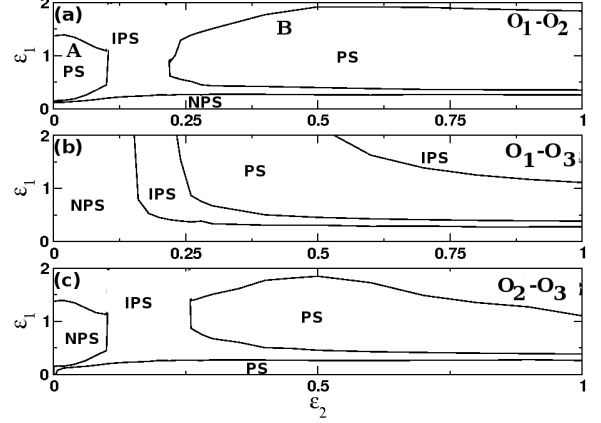


FIG. 2: Schematic phase diagram for the phase synchronization states as a function of ϵ_1 and ϵ_2 between (a) oscillator O_1 and the directly forced response oscillator O_2 , (b) O_1 and the indirectly driven oscillator O_3 , and (c) the mutually coupled driven oscillators O_2 and O_3 . The symbols PS, IPS, and NPS represent the phase synchronization, imperfect phase synchronization, and unsynchronized states respectively.

based upon the time-dependence of the phase difference (see Appendix A). In order to verify the phase synchrony [7, 16] we use the phase for Rössler oscillators in Eq. (1) [17, 18], namely $\phi_i = \arctan(y_i/x_i)$, $i=1, 2, 3$. Phase synchronization (PS) occurs when the phase difference between two interacting oscillators $|\phi_i - \phi_j|$ saturates [16] (see Fig. 2 for details). In imperfect phase synchrony (IPS) regions in Fig. 2, the subsystems are phase locked but are subject to occasional slips. The value of $|\phi_i - \phi_j|$ varies during the evolution of the chaotic system [19], and it changes in a step-wise manner. Each step corresponds to a phase synchronized state under a particular phase locking condition. The jump between consecutive steps occurs in multiples of π [19]. In quasiperiodically forced systems [20] the phase differences in the IPS state also change in arbitrary multiples of π . Fig. 2 shows phase synchronization states between (a) the driving oscillator O_1 and directly driven oscillator O_2 , (b) the driving oscillator O_1 and indirectly driven oscillator O_3 , and (c) mutually coupled response oscillators O_2 and O_3 .

Transitions among the phase synchronization states when the coupling parameters are varied are depicted in Fig. 2(a). In one case unsynchronized oscillators (NPS) become phase synchronized (PS, region B) via the imperfect phase synchronized state (IPS), and in another transition the phase synchronization (PS, region A) transitions to the IPS state. As shown in Fig. 2(b) the indirectly driven oscillator O_3 is initially unsynchronized to the driving oscillator O_1 , but with the increase of mutual interaction between O_2 and O_3 the oscillator O_3 becomes phase synchronized to the drive O_1 . This phase synchro-

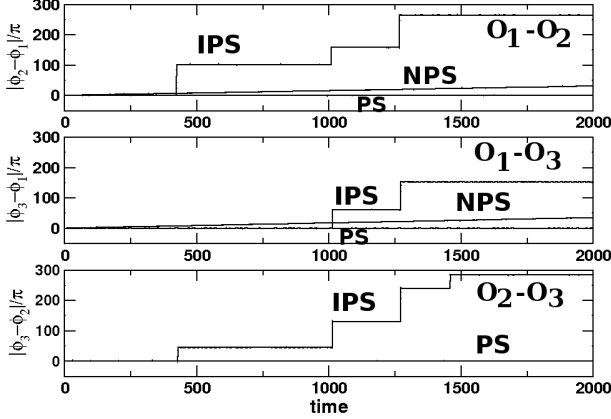


FIG. 3: Phase differences between (a) oscillators O_1 and O_2 for PS at $\epsilon_1 = 1$, IPS at $\epsilon_1=2$, and NPS at $\epsilon_1=0.15$; (b) oscillators O_1 and O_3 for PS at $\epsilon_1=1$, IPS at $\epsilon_1=1.2$, and NPS at $\epsilon_1=0.2$; and (c) oscillators O_2 and O_3 for PS at $\epsilon_1=1$ and IPS at $\epsilon_1=1.5$. The bidirectional coupling parameter is fixed at $\epsilon_2=1$ in all cases.

nization behavior of O_3 clearly signifies the transmission of drive to the mutually coupled oscillators. Phase synchronization between the response subsystems O_2 and O_3 is seen in Fig. 2 (c). By comparing the Figs. 2(a), 2(b), and 2(c) we observe that mutually coupled oscillators O_2 and O_3 are phase synchronized when each of the driven oscillators phase synchronizes with the drive.

The above results are further illustrated at different points in Fig. 3. Fig. 3 (a) describes oscillators O_1 and O_2 , Fig. 3 (b) describes the behavior of O_1 and O_3 , while the phase synchronization states in between the mutually coupled oscillators O_2 and O_3 is shown in Fig. 3 (c). The bidirectional coupling parameter ϵ_2 is kept fixed at $\epsilon_2=1$, and the different curves are drawn for different values of the coupling parameter ϵ_1 . These curves clearly show the behavior of interacting subsystems: the phase difference remains bounded for PS whereas it continuously grows with time in NPS.

We further consider a chain of N oscillators with nearest-neighbor coupling. Phase synchronization between the drive O_1 and responses (O_i , $i = 2, 3, \dots, N$) is lost in such systems as we increase the number of response oscillators. To determine whether the loss of phase synchrony is abrupt or gradual, we first study the case of an extended system consisting of a driving oscillator O_1 and *three* mutually coupled response oscillators ($O_1 \rightarrow O_2 \leftrightarrow O_3 \leftrightarrow O_4$). Among three mutually coupled subsystems, O_2 and O_3 are connected by ϵ_2 while subsystems O_3 and O_4 are connected by the coupling parameter ϵ_3 . Fig. 4 shows the phase synchronization state between the subsystem O_1 and driven subsystems O_2 , O_3 , and O_4 . To observe the effect of forcing in extended systems, the

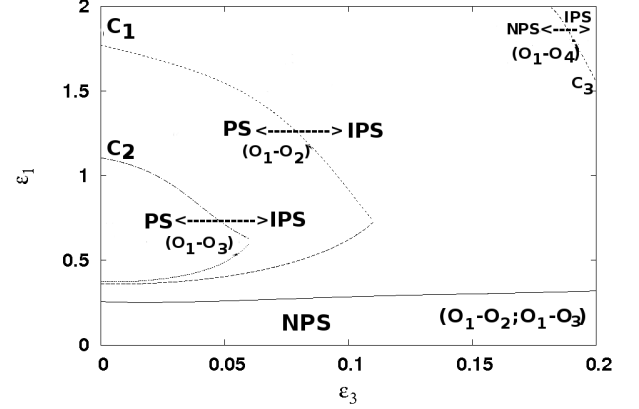


FIG. 4: Phase diagram (schematic) for the coupling parameters ϵ_3 and ϵ_1 . ϵ_3 presents the coupling between O_3 and O_4 . The coupling between O_2 and O_3 is kept fix at $\epsilon_2=1$. C_1 , C_2 , and C_3 are separating curves for the pairs O_1-O_2 , O_1-O_3 , and O_1-O_4 respectively. These curves separate PS, IPS, and NPS for respective pairs. Details are given in text.

coupling between O_2 and O_3 is fixed at $\epsilon_2=1$, while Fig. 4 is for varying ϵ_3 and ϵ_1 .

Our numerical results (see Fig. 4) suggest that phase synchronization is not lost abruptly: with the increase of coupling between O_3 and O_4 the single bounded region of phase synchronization shrinks between O_1-O_2 , O_1-O_3 and the drive and response subsystems turns out to be imperfect phase synchronized (IPS) and then phase unsynchronized (NPS). The phase synchronization states in Fig. 4 are observed for all possible drive-response pairs ($O_1 - O_2$, $O_1 - O_3$, and $O_1 - O_4$), and in comparison of first two pairs we find that the phase space area for phase synchronization (PS) is larger in $O_1 - O_2$ than in $O_1 - O_3$ case. The larger phase space area for PS in $O_1 - O_2$ than in $O_1 - O_3$ shows that in a chain of mutually coupled oscillators, the ability of the drive to synchronize the system decreases with distance from the drive.

It should be noted that curves C_1 and C_2 are boundaries between the phase synchronized (PS) and imperfectly phase synchronized (IPS) region for $O_1 - O_2$ and $O_1 - O_3$ pairs of subsystems respectively (see Appendix A). C_3 separates the phase unsynchronized state (NPS) from the IPS state for the $O_1 - O_4$ pair. The decreasing ability of the drive to cause synchrony is further verified in Fig. 4 where we see that the oscillators O_1 and O_4 are largely phase unsynchronized (NPS) in given parameter range, though we observe the small appearance of IPS state in higher parameter values. The oscillators O_1 and O_4 continue to be in this IPS state for higher values of the varying parameters (ϵ_3 , ϵ_1).

B. Homogeneous Coupling

In this scheme two mutually coupled identical chaotic systems are coupled unidirectionally to another (but non-identical) chaotic oscillator, unidirectionally through the same variable (here y). See Fig. 1(b). The equations of motion for three coupled Rössler oscillators in this scheme are

$$\begin{aligned}
 \dot{x}_1 &= -y_1 - z_1 \\
 \dot{y}_1 &= x_1 + a_1 y_1 \\
 \dot{z}_1 &= b_1 + z_1(x_1 - c_1) \\
 \\
 \dot{x}_2 &= -y_2 - z_2 \\
 \dot{y}_2 &= x_2 + a_2 y_2 + \epsilon_1(y_1 - y_2) + \epsilon_2(y_3 - y_2) \\
 \dot{z}_2 &= b_2 + z_2(x_2 - c_2) \\
 \\
 \dot{x}_3 &= -y_3 - z_3 \\
 \dot{y}_3 &= x_3 + a_2 y_3 + \epsilon_2(y_2 - y_3) \\
 \dot{z}_3 &= b_2 + z_3(x_3 - c_2).
 \end{aligned} \tag{2}$$

A schematic phase diagrams as a function of the parameters ϵ_1 and ϵ_2 is shown in Fig. 5. Fig. 5 (a) shows the phase synchronization states between the driving oscillator O_1 and directly driven oscillator O_2 , while Fig. 5 (b) shows the phase synchronization states between the driving oscillator O_1 and indirectly driven oscillator O_3 . The phase synchronization states between the mutually coupled response subsystems O_2 and O_3 are shown in Fig. 5 (c).

The occurrence of phase synchronization states between O_1 and O_2 or O_3 is a consequence of the transmission of forcing through mutually coupled chaotic oscillators. In order to compare this effect with that of the heterogeneous case (previous subsection) we consider the larger number of oscillators in a chain. As shown in Fig. 6 by increasing the number of oscillators to 4 ($O_1 \rightarrow O_2 \leftrightarrow O_3 \leftrightarrow O_4$) with the coupling between O_2 and O_3 fixed at $\epsilon_2 = 1$ we find that the external drive transmission decreases with distance but the phase synchronization between the drive O_1 and responses O_2 , O_3 , and O_4 continues to occur so long as the coupling is heterogeneous. These results hold for even larger numbers of mutually coupled oscillators. A second difference between hetero- and homogeneous schemes is that there are multiple regimes of PS in the former case (see Fig. 2) but only a single regime for latter.

III. FORCING THROUGH MEDIATING SYSTEMS

Natural systems are often modulated indirectly. Consider a drive-response pair mediated by a number of mutually coupled subsystems. In order to clearly understand the present case of external forcing via intermediate

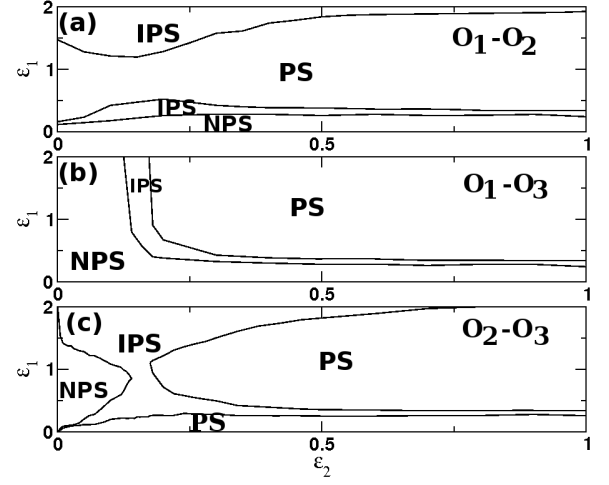


FIG. 5: Schematic phase diagram for the synchronization states as a function of the coupling parameters ϵ_1 and ϵ_2 in the homogeneous coupling scheme between (a) oscillator O_1 and the directly driven oscillator O_2 , (b) O_1 and the indirectly driven oscillator O_3 , and (c) mutually coupled oscillators O_2 and O_3 . The symbols PS, IPS, and NPS have the same meaning as in Fig. 2.

subsystems, the model system is crafted from both heterogeneous and homogeneous coupling schemes. (Recall that the difference between these two coupling schemes is as follows: homogeneous coupling has both the drive and response subsystems coupled by the same variable, while heterogeneous coupling uses different variables.)

The heterogeneous coupling is shown in Fig. 7 (a) where oscillator O_1 drives O_2 and O_3 which then drive oscillator O_4 . If $\epsilon_3 = 0$ this reduces to the case of Fig. 1(a) but for finite ϵ_3 different phase synchronization states (PS, IPS, and NPS) are observed between oscillators O_1 and O_4 . Results are shown in Fig. 7 (b) for a representative value of $\epsilon_2 = 1$. As the number of mutually coupled oscillators between the concerned drive-response pairs (e.g. $O_1 - O_4$) is increased, the PS regime is lost. This result though not presented here, is in consonance with the result of Fig. 4 for the heterogeneous coupling case.

With homogeneous coupling, as shown in Fig. 7 (c), phase synchrony between the terminal oscillators O_1 and O_4 is also achieved. Results are shown in Fig. 7 (d) and it can be seen that for a given value of ϵ_3 oscillators O_1 and O_4 become phase synchronized beyond a critical value of coupling strength ϵ_1 . The non-monotonic form of the border between IPS and PS in Fig. 7 (d) is possibly due to two indirect forcing: O_1 is forcing O_2 and O_3 , while the combined oscillators O_1, O_2 , and O_3 are forcing O_4 . This PS regime persists even in case of increased number of mutually coupled intermediate oscillators.

Heterogeneous and homogeneous coupling schemes thus clearly exhibit distinct dynamics. In the former,

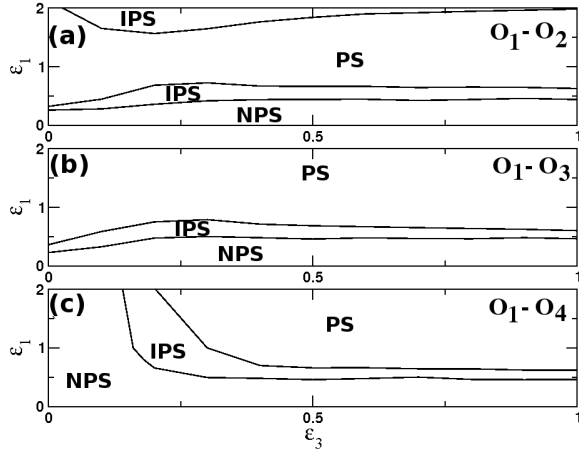


FIG. 6: Schematic phase diagram for the extended case, showing synchronization states as a function of coupling parameters ϵ_3 and ϵ_1 . ϵ_3 is the coupling between O_3 and O_4 , ϵ_1 couples the drive to the responses. Phase synchronization states between (a) drive O_1 and the directly driven oscillator O_2 (b) O_1 and the indirectly driven oscillator O_3 , and (c) O_1 and the indirectly driven oscillator O_4 . In all cases, $\epsilon_2 = 1$.

synchronization regimes are truncated while in the later case synchrony persists even when the number of mutually coupled oscillators is increased. The conjugate variables employed in heterogeneous coupling appear to provide an effective time-delayed interaction [12, 21].

By computing the conditional Lyapunov exponents [22] as a function of the external coupling strength, we find that in general, phase synchronization is not obtained when the drive couples to z , so in the present studies, we drive the variable y (see Eqns. (1) and (2)). Results are similar when drive is coupled with the x variable as well.

IV. SUMMARY

In the present work we have studied the effect of an external drive on symmetrically and diffusively coupled response oscillators. Two different coupling schemes have been examined; these coupling schemes were based on the use of the two coupling parameters either on the same variables (homogeneous coupling) of interacting oscillators or on different variables (heterogeneous coupling). Numerous combinations are possible, and this study is an attempt to consider some representative examples.

The relaying of the drive to both the directly and the indirectly forced subsystems is seen through the occurrence of phase synchronization. For heterogeneous coupling phase synchronization between drive O_1 and responses O_i , $i = 2, 3, \dots, N$ is lost in extended systems, but in a gradual manner. Thus the transmission effect of

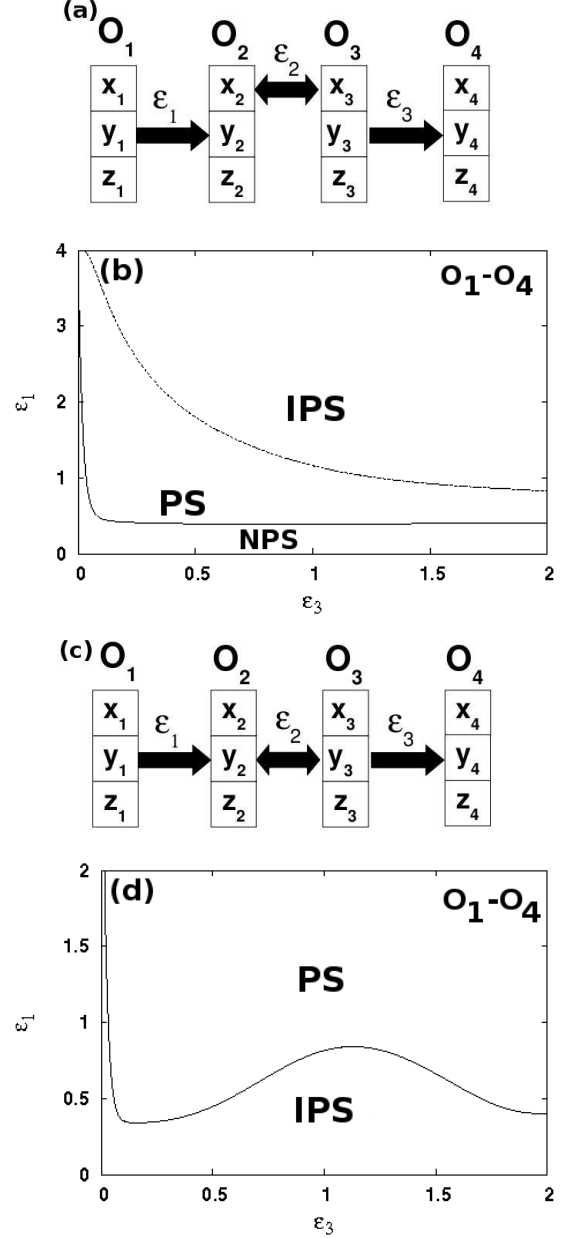


FIG. 7: Heterogeneous coupling: (a) the system of Fig. 1 (a) drives a dissimilar oscillator O_4 through the response oscillator O_3 , (b) a schematic phase diagram as a function of ϵ_3 and ϵ_1 , between oscillators O_1 and O_4 for fixed $\epsilon_2 = 1$. Homogeneous coupling: (c) all oscillators are connected through the same variable, here y , and (d) phase synchronization states between oscillators O_1 and O_4 for fixed $\epsilon_2 = 1$.

external drive decreases sequentially in an array of mutually coupled response oscillators. In case of homogeneous coupling the external drive is transmitted to each of the mutually coupled response subsystems. The effect of indirect external forcing has been also observed for a typical model with mediating oscillators.

The occurrence of phase synchrony is of particular significance due to potential applications in diverse fields

such as biological systems [14], networks of coupled dynamical systems [15] among others. The present results indicate that the transmission of drive to an array of mutually coupled response oscillators depends strongly on the coupling pattern, and it is very likely that the topology of coupling will also play a significant role. A study of different coupling motifs in order to determine the manner in which the drive is transmitted in more complex networks is therefore presently under way [23].

Acknowledgments

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Appendix A: Phase Synchronization

The manner in which the phase difference varies with time distinguishes the states of PS, IPS, and NPS. When two interacting systems are phase synchronized, the phase difference between the systems remain bounded [7] while the phase difference between the interacting subsystems continuously grows when the systems are phase unsynchronized (NPS). In case of imperfect phase synchronization the phase difference grows in multiples of π [19, 20]. This variation in phase difference is used to

deduce a measure for the identification of different phase synchronization states. The averaged phase difference increment Φ is the time average of the derivative of the instantaneous phase difference $\Delta\phi_t = |\phi_2 - \phi_1|$ at time t , namely

$$\Phi = \langle \Delta\phi_{t+1} - \Delta\phi_t \rangle. \quad (A1)$$

1. If the variation of phase difference remains bounded between 0 and π , Φ should be close to zero in case of phase synchronization.
2. For IPS the phase difference between the two interacting subsystems grows in multiples of π ; this results in large values of Φ .
3. When the interacting subsystems are out of phase synchrony, the phase difference increases with time but Φ in considerably lower than it is in the case of IPS.

Fig. 8 shows the averaged phase difference increment Φ as a function of the parameter ϵ_1 for the system in Fig.1 (a) with fixed $\epsilon_2 = 1$. The inset shows the regions of PS, IPS, and NPS. This measure is qualitative in the sense that sharp boundaries between these states (Figs. 2, 4, 5, 6, 7(b) and (d)) cannot be drawn in a quantitative manner. We use thresholds for Φ as 10^{-5} , 0.0001, and 0.1 to assign the boundary for the transition PS to IPS/NPS, NPS to IPS, and NPS to IPS respectively.

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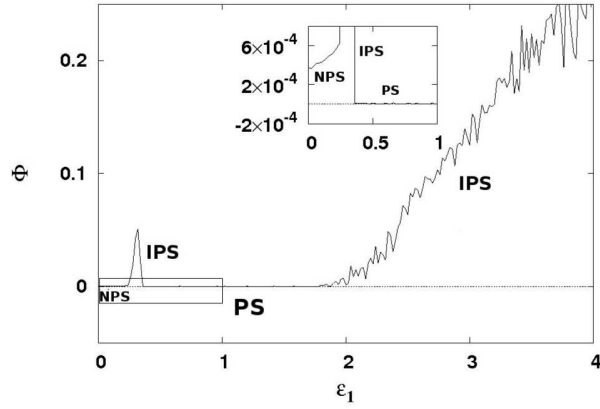


FIG. 8: Phase synchronization measure for *heterogeneous* coupling with $\epsilon_2 = 1$, varying ϵ_1 . The inset shows the order of the averaged phase difference increment in the three different states.

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